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Asymptotic Behavior for Systems of Nonlinear Wave Equations with Multiple Speeds in 3D

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1. Introduction

For the variables $t \in \mathbf{R}$ and $x=(x_1,x_2,x_3) \in \mathbf{R}^3$, we use the notation

$$\partial_0=\partial_t=\frac{\partial}{\partial t},\ \partial_k=\frac{\partial}{\partial x_k} \text{ for } k=1,2,3,$$

and $\Delta_x=\sum_{k=1}^3 \partial_k^2$.

We consider the Cauchy problem for systems of semilinear wave equations of the type

$$(1.1)\quad (\partial_t^2-c_i^2\Delta_x)u_i(t,x)=F_i\bigl(\partial u(t,x)\bigr),\ i=1,\dots,N,\ (t,x)\in(0,\infty)\times\mathbf{R}^3$$

with small initial data

$$(1.2)\qquad u(0,x)=\varepsilon f(x),\ (\partial_t u)(0,x)=\varepsilon g(x),\quad x\in\mathbf{R}^3,$$

where $c_i>0$ for $1\le i\le N$, $u=(u_j)_{1\le j\le N}$, and $\partial u=(\partial_a u_j)_{1\le j\le N,\,0\le a\le 3}$. We suppose that $f,g\in C_0^\infty(\mathbf{R}^3;\mathbf{R}^N)$, and that ε is a small positive parameter. For simplicity we assume that each $F_i=F_i(Y)$ is a homogeneous polynomial of degree p in its arguments, where p is an integer with $p\ge 2$.

We say that the small data global existence (or SDGE in short) holds if for any $f,g\in C_0^\infty(\mathbf{R}^3;\mathbf{R}^N)$ there exists a positive constant ε_0 such that (1.1)-(1.2) admits a unique global solution $u\in C^\infty([0,\infty)\times\mathbf{R}^3;\mathbf{R}^N)$ for any $\varepsilon\in(0,\varepsilon_0]$. It is known that SDGE holds when $p\ge 3$, however SDGE does not hold in general when $p=2$. For example, consider the following single wave equation:

$$(1.3)\qquad (\partial_t^2-c^2\Delta_x)u(t,x)=\bigl(\partial_t u(t,x)\bigr)^2\quad\text{for } (t,x)\in(0,\infty)\times\mathbf{R}^3$$

with initial data (1.2), where c is a positive constant; it is known that there exists a pair of functions $(f,g)\in\bigl(C_0^\infty(\mathbf{R}^3;\mathbf{R})\bigr)^2$ such that the solution u to (1.3) with initial data (1.2) blows up in finite time for any $\varepsilon>0$ (see John [4]; we will give the proof in Section 6 below for the reader's convenience). By contrast, a simple observation due to Nirenberg shows that we have SDGE for a single equation with a special kind of quadratic nonlinearity:

$$(1.4)\quad (\partial_t^2-c^2\Delta_x)u(t,x)=\bigl(\partial_t u(t,x)\bigr)^2-c^2|\nabla_x u(t,x)|^2,\quad (t,x)\in(0,\infty)\times\mathbf{R}^3$$

with initial data (1.2), where $\nabla_x u=(\partial_1 u,\partial_2 u,\partial_3 u)$. If we introduce a new unknown

$$(1.5)\qquad v(t,x)=1-\exp\bigl(-u(t,x)\bigr),$$

then we have $\partial_a v = e^{-u}(\partial_a u)$ and $\partial_a^2 v = e^{-u}\big(\partial_a^2 u - (\partial_a u)^2\big)$ for $0 \le a \le 3$; hence we obtain the Cauchy problem

(1.6) $(\partial_t^2 - c^2\Delta_x)v(t,x) = 0, \qquad (t,x) \in (0,\infty) \times \mathbf{R}^3,$

(1.7) $v(0,x) = 1 - e^{-\varepsilon f(x)},\ (\partial_t v)(0,x) = \varepsilon e^{-\varepsilon f(x)} g(x), \quad x \in \mathbf{R}^3,$

which has a global solution. Conversely, if v is a global solution to (1.6)–(1.7) with $(f,g) \in \big(C_0^\infty(\mathbf{R}^3;\mathbf{R})\big)^2$, and ε is small enough, then we can show that $|v(t,x)| < 1$ for all $(t,x) \in [0,\infty) \times \mathbf{R}^3$; thus

$$u(t,x) = -\log\big(1 - v(t,x)\big), \quad (t,x) \in [0,\infty) \times \mathbf{R}^3$$

is well-defined, and this u is a global solution to (1.4) with initial data (1.2). Hence the case where $p = 2$ is the critical one, and we restrict our consideration to the case $p = 2$ in what follows. In other words, we assume that there are some real constants C_{ijk}^{ab} such that

$$F_i(\partial u) = \sum_{j,k=1}^{N} \sum_{a,b=0}^{3} C_{ijk}^{ab} (\partial_a u_j)(\partial_b u_k). \tag{1.8}$$

For the single speed case where $c_1 = c_2 = \cdots = c_N (= c)$, Klainerman [9] introduced a sufficient condition for SDGE, known as the *null condition* (see also Christodoulou [1]): it is closely related to the example (1.4) above. To state the null condition, we define the *reduced nonlinearity*

$$F_i^{\mathrm{red}}(\omega, X; c) = \sum_{j,k=1}^{N} \sum_{a,b=0}^{3} C_{ijk}^{ab} \omega_a \omega_b X_j X_k$$

for $\omega = (\omega_1, \omega_2, \omega_3) \in S^2$ and $X = (X_1, \ldots, X_N) \in \mathbf{R}^N$ with $\omega_0 = -c$, where the constants C_{ijk}^{ab} are from (1.8). We say that the null condition (for the single speed case where $c_1 = \cdots = c_N = c$) is satisfied if we have

$$F_i^{\mathrm{red}}(\omega, X; c) = 0, \quad \omega \in S^2,\ X \in \mathbf{R}^N,\ i = 1, \ldots, N. \tag{1.9}$$

Let c be a positive constant, and let a and b be integers with $0 \le a < b \le 3$; we introduce the *null forms*

$$Q_0(\varphi, \psi; c) = (\partial_t \varphi)(\partial_t \psi) - c^2 (\nabla_x \varphi) \cdot (\nabla_x \psi), \tag{1.10}$$

$$Q_{ab}(\varphi, \psi) = (\partial_a \varphi)(\partial_b \psi) - (\partial_b \varphi)(\partial_a \psi) \tag{1.11}$$

for smooth functions φ and ψ. Then we can show that the null condition for the single speed case $c_1 = \cdots = c_N = c$ is satisfied if and only if there exist some constants A_{ijk} and B_{ijk}^{ab} such that

$$F_i(\partial u) = \sum_{j,k=1}^{N} A_{ijk} Q_0(u_j, u_k; c) + \sum_{j,k=1}^{N} \sum_{a,b=0}^{3} B_{ijk}^{ab} Q_{ab}(u_j, u_k), \quad i = 1, \ldots, N.$$

Note that the null condition is satisfied for (1.4), however the transformation like (1.5) does not work for systems in general, even if only the null form (1.10) is contained in the nonlinearity.

Concerning the asymptotic behavior of the solution, we can also show that the global solution u under the null condition for the single speed case is *asymptotically free in the energy norm*, that is to say, there exists some free solution $u^+ = (u_j^+)_{1\le j\le N}$ to the system of the free wave equations

$$(\partial_t^2 - c_i^2\Delta_x)u_i^+(t,x) = 0, \quad i = 1,2,\dots,N, \ (t,x) \in (0,\infty)\times \mathbf{R}^3$$

such that

$$\lim_{t\to\infty} \sum_{i=1}^{N} \|(u_i - u_i^+)(t)\|_{E,c_i} = 0, \tag{1.12}$$

where *the energy norm* $\|v(t)\|_{E,c}$ is given by

$$\|v(t)\|_{E,c}^2 = \frac{1}{2}\int_{\mathbf{R}^3} \left(\frac{1}{c^2}|\partial_t v(t,x)|^2 + |\nabla_x v(t,x)|^2\right) dx$$

for a smooth function $v = v(t,x)$ and a positive constant c.

Klainerman's result is extended by Yokoyama [14] to the multiple speed case where the propagation speeds $c_1, c_2, \dots, c_N$ do not necessarily coincide with each other (see also Sideris-Tu [12], and Sogge [13]). Suppose that F_i has the form (1.8). We divide F_i into three kinds of nonlinear terms, depending on the propagation speeds: Let $i = 1,\dots,N$ be given; for $1 \le j,k \le N$ there are two cases $c_j = c_k$, and $c_j \ne c_k$; the case $c_j = c_k$ is further divided into two cases $c_j = c_k = c_i$, and $c_j = c_k \ne c_i$; accordingly we divide F_i as

$$F_i(\partial u) = N_i(\partial u) + R_i^{\mathrm{I}}(\partial u) + R_i^{\mathrm{II}}(\partial u), \quad i = 1,\dots,N, \tag{1.13}$$

where

$$N_i(\partial u) = \sum_{\{j,k;\, c_j=c_k=c_i\}} \sum_{a,b=0}^{3} C_{ijk}^{ab}(\partial_a u_j)(\partial_b u_k),$$

$$R_i^{\mathrm{I}}(\partial u) = \sum_{\{j,k;\, c_j\ne c_k\}} \sum_{a,b=0}^{3} C_{ijk}^{ab}(\partial_a u_j)(\partial_b u_k),$$

$$R_i^{\mathrm{II}}(\partial u) = \sum_{\{j,k;\, c_j=c_k\ne c_i\}} \sum_{a,b=0}^{3} C_{ijk}^{ab}(\partial_a u_j)(\partial_b u_k).$$

We refer to R_i^{I} and R_i^{II} as nonresonant terms of types I and II, respectively. Note that the nonresonant terms of types I and II do not appear in the single speed case. The null condition for the multiple speed case is a restriction on $N_i(\partial u)$: We define

$$N_i^{\mathrm{red}}(\omega, X; c_i) = \sum_{\{j,k;\, c_j=c_k=c_i\}} \sum_{a,b=0}^{3} C_{ijk}^{ab}\omega_a\omega_b X_j X_k$$

for $\omega = (\omega_1, \omega_2, \omega_3) \in S^2$ and $X = (X_1, \dots, X_N) \in \mathbf{R}^N$ with $\omega_0 = -c_i$; we say that the null condition for the multiple speed case is satisfied if we have

$$N_i^{\mathrm{red}}(\omega, X; c_i) = 0, \quad \omega \in S^2,\ X \in \mathbf{R}^N,\ i = 1, \dots, N. \tag{1.14}$$

Similarly to the single speed case, the null condition for the multiple speed case is satisfied if and only if we have

$$\begin{aligned} N_i(\partial u) = & \sum_{\{j,k;\, c_j = c_k = c_i\}} A_{ijk} Q_0(u_j, u_k; c_i) \\ & + \sum_{\{j,k;\, c_j = c_k = c_i\}} B_{ijk}^{ab} Q_{ab}(u_j, u_k), \quad i = 1, \dots, N \end{aligned} \tag{1.15}$$

with some constants A_{ijk} and B_{ijk}^{ab}. Note that (1.14) is a natural extension of the null condition (1.9) for the single speed case. Yokoyama proved SDGE under the null condition for multiple speed case.

If the null condition for the multiple speed case is satisfied, we can show that $N_i(\partial u)$ and $R_i^{\mathrm{I}}(\partial u)$ decay faster than we can expect for general quadratic terms: The interaction between the main parts of the solutions is canceled out for the null forms, while the difference of the propagation speeds makes the interaction of the main parts weaker for the nonresonant terms of type I. Especially we have

$$N_i(\partial u) + R_i^{\mathrm{I}}(\partial u) \in L^1\big((0, \infty); L^2(\mathbf{R}^3)\big) \tag{1.16}$$

under the null condition (see Lemma 4.4 below). Making use of (1.16), one can easily show that the solution is asymptotically free if $R_i^{\mathrm{II}} \equiv 0$ for all $i = 1, \dots, N$ (see Lemma 2.1 below for the details). The situation is different for the nonresonant terms of type II. We cannot expect that the nonresonant terms of type II decay faster than general quadratic terms; we have $R_i^{\mathrm{II}}(\partial u) \notin L^1\big((0, \infty); L^2(\mathbf{R}^3)\big)$ in general, and it is not clear whether the solution is asymptotically free or not when $R_i^{\mathrm{II}} \not\equiv 0$. We have to analyze the interaction between the fundamental solution and R_i^{II} in order to understand the effect of the nonresonant terms of type II. For this purpose we will investigate the asymptotic pointwise behavior of the solutions.

Before we proceed to the main results on the asymptotic behavior for the system of semilinear wave equations, we review the known results for the linear wave equations in the next section.

Throughout this paper, various positive constants, which may change line by line, are denoted by the same letter C.

2. Asymptotic Behavior for Linear Wave Equations

Let $c > 0$, and we consider the following Cauchy problem for the single free wave equation with the propagation speed c:

$$(\partial_t^2 - c^2 \Delta_x) v(t, x) = 0, \qquad (t, x) \in \mathbf{R} \times \mathbf{R}^3, \tag{2.1}$$

$$v(0, x) = \varphi(x),\ (\partial_t v)(0, x) = \psi(x), \qquad x \in \mathbf{R}^3. \tag{2.2}$$

If $(\varphi, \psi) \in C_0^\infty(\mathbf{R}^3) \times C_0^\infty(\mathbf{R}^3)$, it is well known that the solution v can be written as

$$v(t,x) = \mathcal{U}_0[\varphi, c^{-1}\psi](ct, x) \tag{2.3}$$

for $(t,x) \in (0,\infty) \times \mathbf{R}^3$, where $\mathcal{U}_0$ is given by

$$\mathcal{U}_0[\varphi,\psi](t,x) := \frac{1}{4\pi t}\int_{|y-x|=t} \psi(y)dS_y + \partial_t\left(\frac{1}{4\pi t}\int_{|y-x|=t} \varphi(y)dS_y\right) \tag{2.4}$$

for $(t,x) \in (0,\infty) \times \mathbf{R}^3$. Here dS_y denotes the area element on the sphere of radius t centered at x. For $(t,x) \in (-\infty, 0) \times \mathbf{R}^3$, we have

$$v(t,x) = \mathcal{U}_0[\varphi, -c^{-1}\psi](-ct, x).$$

From these expressions, we see that

$$\operatorname{supp}\varphi \cup \operatorname{supp}\psi \subset B_R := \{x \in \mathbf{R}^3; |x| \le R\}$$

with $R > 0$ implies

$$v(t,x) = 0, \quad \big||x| - c|t|\big| \ge R, \ t \in \mathbf{R}. \tag{2.5}$$

This property is called the *Huygens principle.*

For $c > 0$ and $t \in \mathbf{R}$, we define the mapping $S_c(t)$ by

$$S_c(t) : C_0^\infty(\mathbf{R}^3) \times C_0^\infty(\mathbf{R}^3) \ni (\varphi,\psi) \mapsto (v(t,\cdot), \partial_t v(t,\cdot)) \in C_0^\infty(\mathbf{R}^3) \times C_0^\infty(\mathbf{R}^3),$$

where v is the solution to (2.1)–(2.2). Then we have the conservation of the energy

$$\|S_c(t)(\varphi,\psi)\|_{H_c} \left(= \big\|(v(t,\cdot), \partial_t v(t,\cdot))\big\|_{H_c}\right) = \|(\varphi,\psi)\|_{H_c}, \quad t \in \mathbf{R}, \tag{2.6}$$

where $\|\cdot\|_{H_c}$ is defined by

$$\begin{aligned}\|(\varphi,\psi)\|_{H_c}^2 &= \frac{1}{2}\int_{\mathbf{R}^3}\left(|\nabla_x\varphi(x)|^2 + \frac{1}{c^2}|\psi(x)|^2\right)dx \\ &= \frac{1}{2}\left(\|\varphi\|_{\dot H^1(\mathbf{R}^3)}^2 + \frac{1}{c^2}\|\psi\|_{L^2(\mathbf{R}^3)}^2\right).\end{aligned}$$

Here $\dot H^1(\mathbf{R}^3)$ denotes the homogeneous Sobolev space. Note that $\|v(t,\cdot)\|_{E,c} = \big\|(v(t,\cdot), \partial_t v(t,\cdot))\big\|_{H_c}$, where $\|\cdot\|_{E,c}$ is the energy norm defined in Section 1. By the uniqueness of the solution, we get

$$S_c(t+s) = S_c(t)S_c(s), \quad t,s \in \mathbf{R}.$$

Let H_c be the completion of $C_0^\infty(\mathbf{R}^3) \times C_0^\infty(\mathbf{R}^3)$ with respect to the norm $\|\cdot\|_{H_c}$. Then for each $t \in \mathbf{R}$, $S_c(t)$ can be uniquely extended to a unitary operator on H_c, and we write $S_c(t)$ also for this extended operator. Observe that we have $(\varphi,\psi) \in H_c$ if and only if $(\varphi,\psi) \in \dot H^1(\mathbf{R}^3) \times L^2(\mathbf{R}^3)$. For $(\varphi,\psi) \in H_c$, if we write $\big(v(t,\cdot), (\partial_t v)(t,\cdot)\big) = S_c(t)(\varphi,\psi)$, v gives the solution to (2.1)–(2.2) with $(v, \partial_t v) \in C\big(\mathbf{R}; \dot H^1(\mathbf{R}^3)\big) \times C\big(\mathbf{R}; L^2(\mathbf{R}^3)\big)$.

Let $(\varphi, \psi) \in \dot{H}^1(\mathbf{R}^3) \times L^2(\mathbf{R}^3)$, and let $G \in L^1\big((0,T); L^2(\mathbf{R}^3)\big)$ for any $T > 0$. We consider the Cauchy problem for the inhomogeneous wave equation

$$(\partial_t^2 - c^2\Delta_x)w(t,x) = G(t,x), \qquad (t,x) \in (0,\infty) \times \mathbf{R}^3, \tag{2.7}$$

$$w(0,x) = \varphi(x),\ (\partial_t w)(0,x) = \psi(x), \qquad x \in \mathbf{R}^3. \tag{2.8}$$

Then by the Duhamel principle, we get

$$\big(w(t,\cdot), \partial_t w(t,\cdot)\big) = S_c(t)(\varphi,\psi) + \int_0^t S_c(t-\tau)\big(0, G(\tau,\cdot)\big)d\tau, \quad t \geq 0. \tag{2.9}$$

Lemma 2.1. *If $G \in L^1\big((0,\infty); L^2(\mathbf{R}^3)\big)$, then the solution w to (2.7)–(2.8) is asymptotically free in the energy norm; namely there is $(\varphi^+, \psi^+) \in \dot{H}^1(\mathbf{R}^3) \times L^2(\mathbf{R}^3)$ such that*

$$\lim_{t\to\infty} \|w(t) - w^+(t)\|_{E,c} \left(= \lim_{t\to\infty} \big\|\big(w(t), \partial_t w(t)\big) - \big(w^+(t), \partial_t w^+(t)\big)\big\|_{H_c}\right) = 0,$$

where $\big(w^+(t), \partial_t w^+(t)\big) = S_c(t)(\varphi^+, \psi^+)$, namely w^+ is the solution to the free wave equation $(\partial_t^2 - c^2\Delta_x)w^+ = 0$ with initial data $(w^+, \partial_t w^+) = (\varphi^+, \psi^+)$.

Proof. Since we have

$$\begin{aligned}\left\|\int_0^\infty S_c(-\tau)\big(0, G(\tau)\big)d\tau\right\|_{H_c} &\leq \int_0^\infty \big\|S_c(-\tau)\big(0, G(\tau)\big)\big\|_{H_c} d\tau \\ &= \frac{1}{\sqrt{2}c}\int_0^\infty \|G(\tau)\|_{L^2(\mathbf{R}^3)}\, d\tau < \infty,\end{aligned}$$

it follows that

$$(\varphi^+, \psi^+) = (\varphi, \psi) + \int_0^\infty S_c(-\tau)\big(0, G(\tau)\big)d\tau \in H_c$$

is well-defined. Let $\big(w^+(t), \partial_t w^+(t)\big) = S_c(t)(\varphi^+, \psi^+)$. Then we obtain

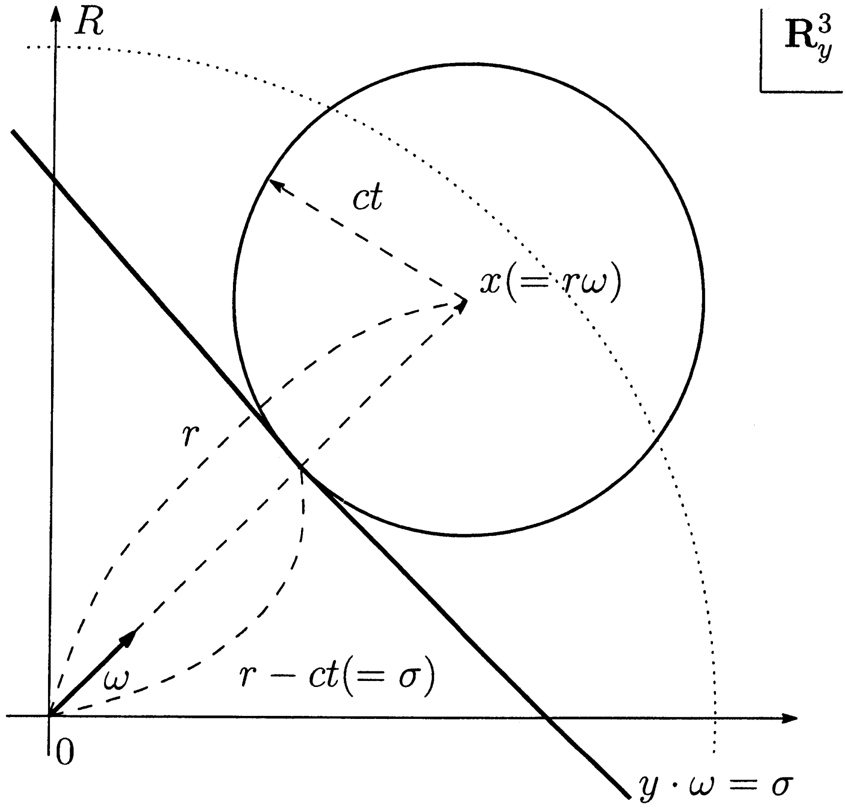
$$\begin{aligned}&\big\|\big(w(t), \partial_t w(t)\big) - \big(w^+(t), \partial_t w^+(t)\big)\big\|_{H_c} \\ &= \left\|\int_t^\infty S_c(t-\tau)\big(0, G(\tau)\big)d\tau\right\|_{H_c} \leq \int_t^\infty \big\|S_c(t-\tau)\big(0, G(\tau)\big)\big\|_{H_c} d\tau \\ &= \frac{1}{\sqrt{2}c}\int_t^\infty \|G(\tau)\|_{L^2(\mathbf{R}^3)}\, d\tau \to 0 \quad (t \to \infty).\end{aligned}$$

This completes the proof. □

Now we turn our attention to the asymptotic pointwise behavior of the solution v to (2.1)–(2.2) with $(\varphi, \psi) \in C_0^\infty(\mathbf{R}^3) \times C_0^\infty(\mathbf{R}^3)$. If we look at the solution for large t with fixed x, it just vanishes after a finite time because of the Huygens principle. Hence it is more reasonable to look at the solution for large t (or large $|x|$) with $|x| - ct$ being fixed. For a compactly supported function g, we can easily expect that we have

$$\lim_{t\to\infty}\left(\int_{|y-x|=ct} g(y)dS_y\right)\Bigg|_{|x|=r,\ x/|x|=\omega,\ r-ct=\sigma} = \int_{y\cdot\omega=\sigma} g(y)dS_y'$$

with $\sigma \in \mathbf{R}$ and $\omega \in S^2$ being fixed (see the figure below), where dS'_y is the area element on the plane $\{y \in \mathbf{R}^3; y \cdot \omega = \sigma\}$.



In this way, taking the expression (2.3) and (2.4) into account, we are led to the *Friedlander radiation field.* For $\varphi, \psi \in C_0^\infty(\mathbf{R}^3)$, we define the Friedlander radiation field

$$\mathcal{F}_0[\varphi,\psi](\sigma,\omega) = \frac{1}{4\pi}\left(\mathcal{R}[\psi](\sigma,\omega) - (\partial_\sigma \mathcal{R}[\varphi])(\sigma,\omega)\right), \quad (\sigma,\omega) \in \mathbf{R} \times S^2, \tag{2.10}$$

where $\mathcal{R}[v]$ denotes the *Radon transform* of v, i.e.,

$$\mathcal{R}[v](\sigma,\omega) = \int_{y\cdot\omega=\sigma} v(y)dS'_y.$$

Note that $\operatorname{supp}\varphi \cup \operatorname{supp}\psi \subset B_R$ implies

$$\mathcal{F}_0[\varphi,\psi](\sigma,\omega) = 0, \quad |\sigma| \geq R, \ \omega \in S^2. \tag{2.11}$$

We also have $\mathcal{F}_0[\varphi,\psi] \in C^\infty(\mathbf{R} \times S^2)$.

Lemma 2.2. *Let $c > 0$ and $0 < c_0 < c$. Then for any $(\varphi,\psi) \in C_0^\infty(\mathbf{R}^3) \times C_0^\infty(\mathbf{R}^3)$ and any $\nu \geq 0$ there exists a positive constant C such that*

$$\begin{aligned} &|rv(t,r\omega) - V(r-ct,\omega)| + |r(\partial_t v)(t,r\omega) - (-c)(\partial_\sigma V)(r-ct,\omega)| \\ &\quad + \sum_{k=1}^{3} |r(\partial_k v)(t,r\omega) - \omega_k(\partial_\sigma V)(r-ct,\omega)| \\ &\qquad \leq C(1+t+r)^{-1}(1+|ct-r|)^{-\nu} \end{aligned} \tag{2.12}$$

for $r \geq c_0 t \geq 1$ and $\omega = (\omega_1, \omega_2, \omega_3) \in S^2$, where

$$\big(v(t,\cdot), \partial_t v(t,\cdot)\big) = S_c(t)(\varphi, \psi), \ V(\sigma,\omega) = \mathcal{F}_0[\varphi, c^{-1}\psi](\sigma,\omega).$$

Proof. In view of the Huygens principle and (2.11), $(1+|ct-r|)$ is bounded in the support of the functions on the left-hand side of (2.12), hence it suffices to prove the result for $\nu = 0$. The proof for the case where $c=1$, $c_0 = 1/2$, and $\nu = 0$ can be found in Hörmander [3] for instance (see also Katayama-Kubo [6]), and we only need some apparent modification to obtain the general case. □

For $(\varphi, \psi) \in \dot{H}^1(\mathbf{R}^3) \times L^2(\mathbf{R}^3)$, we do not have the asymptotic pointwise behavior like Lemma 2.2, but the Friedlander radiation field still works through the *translation representation* of the solution for such initial data. We define the mapping T_c by

$$(2.13) \qquad T_c : C_0^\infty(\mathbf{R}^3) \times C_0^\infty(\mathbf{R}^3) \ni (\varphi,\psi) \mapsto \partial_\sigma \mathcal{F}_0[\varphi, c^{-1}\psi] \in C_0^\infty(\mathbf{R}\times S^2),$$

namely we define

$$\big(T_c(\varphi,\psi)\big)(\sigma,\omega) = \partial_\sigma \mathcal{F}_0[\varphi, c^{-1}\psi](\sigma,\omega), \quad (\sigma,\omega) \in \mathbf{R}\times S^2.$$

T_c is called the translation representation because we have

$$\big(T_c S_c(t)(\varphi,\psi)\big)(\sigma,\omega) = \big(T_c(\varphi,\psi)\big)(\sigma - ct, \omega), \quad (\sigma,\omega) \in \mathbf{R}\times S^2, \ t \in \mathbf{R}.$$

It is known that we have

$$\|T_c(\varphi,\psi)\|_{L^2(\mathbf{R}\times S^2)} = \|(\varphi,\psi)\|_{H_c}, \quad (\varphi,\psi) \in C_0^\infty(\mathbf{R}^3)\times C_0^\infty(\mathbf{R}^3),$$

and that T_c can be uniquely extended to an isometric isomorphism from H_c onto $L^2(\mathbf{R}\times S^2)$ (see Lax-Phillips [10, Chapter IV]); we write T_c also for this extension.

For a real-valued function $W \in L^2(\mathbf{R}\times S^2)$, we define an $\mathbf{R}^4$-valued function $D_c W$ by

$$(2.14) \qquad (D_c W)(t,x) = \big(|x|^{-1} W(|x| - ct, x/|x|)\big)(-1, x_1/|x|, x_2/|x|, x_3/|x|)$$

for $(t,x) \in [0,\infty) \times (\mathbf{R}^3 \setminus \{0\})$. Note that we have

$$\begin{aligned}\int_{\mathbf{R}^3} |(D_c W)(t,x)|^2 dx =& 2\int_0^\infty \left(\int_{S^2} |W(r-ct,\omega)|^2 dS_\omega\right) dr \\ =& 2\int_{-ct}^\infty \left(\int_{S^2} |W(\sigma,\omega)|^2 dS_\omega\right) d\sigma \leq 2\|W\|^2_{L^2(\mathbf{R}\times S^2)},\end{aligned}$$

where dS_ω is the area element on S^2. The following lemma is an essential tool to combine the asymptotic pointwise behavior to the asymptotic freedom in the energy norm:

Lemma 2.3. *Assume that $W \in L^2(\mathbf{R}\times S^2)$ be given. We put*

$$\begin{aligned}&(\varphi,\psi) := T_c^{-1} W \in H_c, \\ &\big(v(t,\cdot), \partial_t v(t,\cdot)\big) := S_c(t)(\varphi,\psi) (= S_c(t) T_c^{-1} W) \in H_c.\end{aligned}$$

Then we have

$$\lim_{t\to\infty} \left\| D_c W(t,\cdot) - \left(c^{-1}\partial_t v(t,\cdot), \nabla_x v(t,\cdot)\right) \right\|_{L^2(\mathbf{R}^3)} = 0.$$

Proof. Let $\varepsilon > 0$. Then there is $(\varphi_0, \psi_0) \in C_0^\infty(\mathbf{R}^3) \times C_0^\infty(\mathbf{R}^3)$ such that

$$\|(\varphi, \psi) - (\varphi_0, \psi_0)\|_{H_c} < \varepsilon.$$

We put $\left(v_0(t,\cdot), \partial_t v_0(t,\cdot)\right) = S_c(t)(\varphi_0, \psi_0)$, and $W_0 = T_c(\varphi_0, \psi_0)$. Then we get

$$\|(v - v_0)(t)\|_{E,c} = \|W - W_0\|_{L^2(\mathbf{R}\times S^2)} = \|(\varphi, \psi) - (\varphi_0, \psi_0)\|_{H_c} < \varepsilon.$$

It follows that

$$\begin{aligned}
&\left\| D_c W(t,\cdot) - \left(c^{-1}\partial_t v(t,\cdot), \nabla_x v(t,\cdot)\right) \right\|_{L^2(\mathbf{R}^3)} \\
&\quad \le \left\| D_c W_0(t,\cdot) - \left(c^{-1}\partial_t v_0(t,\cdot), \nabla_x v_0(t,\cdot)\right) \right\|_{L^2(\mathbf{R}^3)} \\
&\qquad + \sqrt{2}\|(v - v_0)(t)\|_{E,c} + \sqrt{2}\|W - W_0\|_{L^2(\mathbf{R}\times S^2)} \\
&\quad \le \left\| D_c W_0(t,\cdot) - \left(c^{-1}\partial_t v_0(t,\cdot), \nabla_x v_0(t,\cdot)\right) \right\|_{L^2(\mathbf{R}^3)} + 2\sqrt{2}\varepsilon.
\end{aligned}$$

Thus our task is to prove

$$\lim_{t\to\infty} \left\| D_c W_0(t,\cdot) - \left(c^{-1}\partial_t v_0(t,\cdot), \nabla_x v_0(t,\cdot)\right) \right\|_{L^2(\mathbf{R}^3)} = 0. \tag{2.15}$$

Indeed, once (2.15) is established, we get

$$\limsup_{t\to\infty} \left\| D_c W(t,\cdot) - \left(c^{-1}\partial_t v(t,\cdot), \nabla_x v(t,\cdot)\right) \right\|_{L^2(\mathbf{R}^3)} \le 2\sqrt{2}\varepsilon$$

for any $\varepsilon > 0$, which implies the desired result immediately.

We choose $0 < c_0 < c$. Since (φ_0, ψ_0) is compactly supported, there is a positive constant R such that $\operatorname{supp}\varphi_0 \cup \operatorname{supp}\psi_0 \subset B_R$. Then the Huygens principle and (2.11) yield

$$\left(c^{-1}\partial_t v_0(t,x), \nabla_x v_0(t,x)\right) = D_c W_0(t,x) = 0$$

for $(t,x) \in [0,\infty) \times \mathbf{R}^3$ with $|x| \le ct - R$. Observing that we have $c_0 t \le ct - R$ for $t \ge R/(c - c_0)$, we obtain

$$\left(\int_{|x|\le c_0 t} \left| D_c W_0(t,x) - \left(c^{-1}\partial_t v_0(t,x), \nabla_x v_0(t,x)\right) \right|^2 dx \right)^{1/2} = 0 \tag{2.16}$$

for $t \ge R/(c - c_0)$.

By Lemma 2.2, we get

$$\left| D_c W_0(t,x) - \left(c^{-1}\partial_t v_0(t,x), \nabla_x v_0(t,x)\right) \right|^2 \le C(1 + t + |x|)^{-2}|x|^{-2}$$

for $|x| \ge c_0 t \ge 1$, which leads to

$$\begin{aligned}
&\left(\int_{|x|\ge c_0 t} \left| D_c W_0(t,x) - \left(c^{-1}\partial_t v_0(t,x), \nabla_x v_0(t,x)\right) \right|^2 dx \right)^{1/2} \\
&\quad \le C\left(\int_0^\infty (1 + t + r)^{-2} dr \right)^{1/2} \le C(1+t)^{-1/2} \to 0 \quad (t \to \infty).
\end{aligned} \tag{2.17}$$

We obtain (2.15) from (2.16) and (2.17). This completes the proof. $\square$

Let $u=(u_1,\dots,u_N)$ be the solution to (1.1)–(1.2). From Lemma 2.3, we see that if we can find $V_i=V_i(\sigma,\omega)\in L^2(\mathbf{R}\times S^2)$ such that

$$\lim_{t\to\infty}\sum_{i=1}^{N}\left\|\left(c_i^{-1}\partial_t u_i(t,\cdot),\nabla_x u_i(t,\cdot)\right)-\varepsilon D_{c_i}V_i(t,\cdot)\right\|_{L^2(\mathbf{R}^3)}=0, \tag{2.18}$$

then we obtain

$$\lim_{t\to\infty}\sum_{i=1}^{N}\|u_i(t)-u_i^+(t)\|_{E,c_i}=0,$$

where $\left(u_i^+(t,\cdot),\partial_t u_i^+(t,\cdot)\right)=\varepsilon S_{c_i}(t)T_{c_i}^{-1}V_i$. Our strategy here is to obtain (2.18) by deriving asymptotic pointwise behavior, which is similar to that in Lemma 2.2, for systems of semilinear wave equations.

3. The Main Results

The first result is the asymptotic pointwise behavior for the general case under the null condition for the multiple speed case.

Theorem 3.1. *Fix $0<\delta\ll 1$. Let $0<c_0\ll 1$ so that $c_0\le\min\{c_1,\dots,c_N\}/2$. Suppose that the null condition* (1.14) *for the multiple speed case is satisfied. Let $u=(u_j)_{1\le j\le N}$ be the global solution to* (1.1)–(1.2). *Then for any $f,g\in C_0^\infty(\mathbf{R}^3;\mathbf{R}^N)$ and sufficiently small $\varepsilon>0$, there exist a function $V=(V_i)_{1\le i\le N}$ of $(\sigma,\omega)\in\mathbf{R}\times S^2$, and a positive constant C such that*

$$\begin{aligned}&|r(\partial_t u_i)(t,r\omega)-(-c_i)\varepsilon V_i(r-c_it,\omega)|\\&\quad+\sum_{k=1}^{3}|r(\partial_k u_i)(t,r\omega)-\omega_k\varepsilon V_i(r-c_it,\omega)|\\&\quad\le C\varepsilon(1+t+r)^{-1+\delta}(1+|c_it-r|)^{-\delta},\quad 1\le i\le N\end{aligned}\tag{3.1}$$

for $r\ge c_0t\ge 1$ and $\omega=(\omega_1,\omega_2,\omega_3)\in S^2$. Moreover we have

$$\left|V_i(\sigma,\omega)-\left(\partial_\sigma U_i^0\right)(\sigma,\omega)\right|\le C\varepsilon(1+|\sigma|)^{-1},\quad 1\le i\le N \tag{3.2}$$

for $(\sigma,\omega)\in\mathbf{R}\times S^2$, where $U_i^0=\mathcal{F}_0[f_i,g_i]$. Here V may depend on ε, but C is independent of ε.

Comparing (3.1) with (2.12), we see that ∂u_i asymptotically behaves like derivatives of free solutions in the pointwise sense even if $R_i^{\mathrm{II}}\not\equiv 0$. Since U_i^0 vanishes for large $|\sigma|$ (see (2.11)), (3.2) implies that $V_i\in L^2(\mathbf{R}\times S^2)$. Moreover we have

$$\lim_{t\to\infty}\left(\int_{|x|\le c_0t}|D_{c_i}V_i(t,x)|^2dx\right)^{1/2}=0. \tag{3.3}$$

From the *a priori* estimate for u (see Lemma 4.3 below), we get

$$\lim_{t\to\infty}\left(\int_{|x|\le c_0t}\left|\left(c_i^{-1}\partial_t u_i(t,x),\nabla_x u_i(t,x)\right)\right|^2dx\right)^{1/2}=0. \tag{3.4}$$

By (3.1), we obtain

$$\left(\int_{|x|\geq c_0 t} \left|\left(c_i^{-1}\partial_t u(t,x), \nabla_x u(t,x)\right) - \varepsilon D_{c_i} V_i(t,x)\right|^2 dx\right)^{1/2} \leq C\varepsilon(1+t)^{-(1/2)+\delta} \to 0 \quad (t\to\infty). \tag{3.5}$$

From (3.3), (3.4), and (3.5) we obtain (2.18), which leads to the following:

Corollary 3.2. *Suppose that the null condition* (1.14) *for the multiple speed case is satisfied. Then, for any $f, g \in C_0^\infty(\mathbf{R}^3; \mathbf{R}^N)$ and sufficiently small $\varepsilon > 0$, there exist $f^+ = (f_j^+)_{1\leq j\leq N} \in \dot{H}^1(\mathbf{R}^3; \mathbf{R}^N)$ and $g^+ = (g_j^+)_{1\leq j\leq N} \in L^2(\mathbf{R}^3; \mathbf{R}^N)$ such that*

$$\lim_{t\to\infty} \sum_{i=1}^N \|(u_i - u_i^+)(t)\|_{E,c_i} = 0,$$

where $u = (u_j)_{1\leq j\leq N}$ is the global solution to (1.1)–(1.2), *and each u_i^+ is the solution to $(\partial_t^2 - c_i^2\Delta_x)u_i^+ = 0$ with initial data $u_i^+ = f_i^+$ and $\partial_t u_i^+ = g_i^+$ at $t = 0$.*

Hence the global solution to (1.1) with small data is asymptotically free in the energy norm even if the nonresonant terms of type II are present in the nonlinearity.

Next we will see that we have a better asymptotic pointwise behavior if the nonresonant terms of type II are not included in the nonlinearity.

Theorem 3.3. *Let $0 < \rho < 1$, and let c_0 and $u = (u_j)_{1\leq j\leq N}$ be as in Theorem* 3.1. *Suppose that the null condition* (1.14) *for the multiple speed case is satisfied. If we assume $R_i^{\mathrm{II}} \equiv 0$ for $1 \leq i \leq N$ in addition, then for any $f, g \in C_0^\infty(\mathbf{R}^3; \mathbf{R}^N)$ and sufficiently small $\varepsilon > 0$, there exist a function $U = (U_i)_{1\leq i\leq N}$ of $(\sigma,\omega) \in \mathbf{R}\times S^2$, and a positive constant C such that*

$$|ru_i(t, r\omega) - \varepsilon U_i(r - c_i t, \omega)| \leq C\varepsilon(1+t+r)^{-\rho}, \quad 1 \leq i \leq N, \tag{3.6}$$

$$\begin{aligned} &|r(\partial_t u_i)(t, r\omega) - (-c_i)\varepsilon(\partial_\sigma U_i)(r - c_i t, \omega)| \\ &\quad + \sum_{k=1}^3 |r(\partial_k u_i)(t, r\omega) - \omega_k\varepsilon(\partial_\sigma U_i)(r - c_i t, \omega)| \\ &\qquad \leq C\varepsilon(1+t+r)^{-1}(1+|c_i t - r|)^{-\rho}, \quad 1 \leq i \leq N \end{aligned} \tag{3.7}$$

for $r \geq c_0 t \geq 1$ and $\omega = (\omega_1, \omega_2, \omega_3) \in S^2$. Moreover we have

$$\sum_{i=1}^N \left|(\partial_\sigma^k U_i)(\sigma,\omega) - \left(\partial_\sigma^k U_i^0\right)(\sigma,\omega)\right| \leq C\varepsilon(1+|\sigma|)^{-k-\rho} \tag{3.8}$$

for $(\sigma,\omega) \in \mathbf{R}\times S^2$ and $k = 0, 1$, where U_i^0 is defined as in Theorem 3.1.

Comparing (3.7) with (3.1), the convergence rate is improved; more importantly (3.6) gives the asymptotic pointwise behavior of $u = (u_1, \ldots, u_N)$ itself. We see that not only ∂u_i but also u_i behaves similarly to the free solutions when $R_i^{\mathrm{II}} \equiv 0$ for $1 \leq i \leq N$.

Now we are led to the question whether we can obtain an estimate corresponding to (3.6) when the nonresonant terms of type II are included in the nonlinearity, and u itself behaves similarly to the free solution: The next result shows that this is impossible, and that u behaves differently from free solutions in some case where $R_i^{\mathrm{II}} \not\equiv 0$; in other words, the assumption $R_i^{\mathrm{II}} \equiv 0$ for $1 \le i \le N$ is essential to derive (3.6) in Theorem 3.3.

Theorem 3.4. *Let $0 < c_2 < c_1$. Suppose that $u = (u_1, u_2)$ be the global solution to*

$$\begin{cases} (\partial_t^2 - c_1^2\Delta_x)u_1 = A(\partial_t u_2)^2, \\ (\partial_t^2 - c_2^2\Delta_x)u_2 = (\partial_t u_1)^2, \end{cases} \qquad (t,x) \in (0,\infty) \times \mathbf{R}^3 \tag{3.9}$$

with initial data $u = \varepsilon f$ and $\partial_t u = \varepsilon g$ at $t = 0$, where A is a nonnegative constant. Then there exist $f, g \in C_0^\infty(\mathbf{R}^3; \mathbf{R}^2)$, $R > 0$, $T_0 > 0$, and $C > 0$ such that

$$C^{-1}\varepsilon\left(1 + \varepsilon \log(2+t)\right) \le |ru_2(t,x)| \le C\varepsilon\left(1 + \varepsilon\log(2+t)\right) \tag{3.10}$$

for $T_0 \le c_2 t \le |x| \le c_2 t + R$, provided that ε is small enough.

This estimate shows that u_2 decays slightly slower than the free solutions. More precisely, (3.6) implies

$$u_i(t,x) \sim C\varepsilon(1+t+|x|)^{-1}$$

along the line $\{|x| - c_i t = \sigma, x/|x| = \omega\}$ for large t and fixed $(\sigma,\omega) \in \mathbf{R} \times S^2$, while (3.10) leads to

$$u_2(t,x) \sim C\varepsilon(1+t+|x|)^{-1}\left(1+\varepsilon\log(2+t)\right)$$

along the line $\{|x| - c_2 t = \sigma, x/|x| = \omega\}$ for large t and fixed $(\sigma,\omega) \in [0,R] \times S^2$.

In conclusion, the effect of the nonresonant terms of type II is so weak that the solution u exists globally, its derivatives behave similarly to those of the free solutions, and the solution u is asymptotically free in the energy norm; but it is strong enough to affect the decay rate of the solution u itself.

4. Preliminary Results

For $c > 0$ we put $\Box_c = \partial_t^2 - c^2\Delta_x$. We introduce

$$\begin{aligned} S &= t\partial_t + x \cdot \nabla_x, \\ \Omega &= (\Omega_1, \Omega_2, \Omega_3) = x \times \nabla_x = (x_2\partial_3 - x_3\partial_2, x_3\partial_1 - x_1\partial_3, x_1\partial_2 - x_2\partial_1), \end{aligned}$$

where the symbols "$\cdot$" and "$\times$" denote the inner and the exterior products in $\mathbf{R}^3$, respectively. We put

$$\Gamma = (\Gamma_0, \Gamma_1, \dots, \Gamma_7) = (S, \Omega, \partial)$$

with $\partial = (\partial_0, \partial_1, \partial_2, \partial_3)$. We write $\Gamma^\alpha = \Gamma_0^{\alpha_0}\Gamma_1^{\alpha_1}\cdots\Gamma_7^{\alpha_7}$ with a multi-index $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_7)$. Since we have $[S, \Box_c] = -2\Box_c$, and $[\partial_a, \Box_c] = [\Omega_j, \Box_c] = 0$ for $0 \le a \le 3$ and $1 \le j \le 3$, we have

$$\Gamma^\alpha\varphi(t,x) = (\Gamma_0 + 2)^{\alpha_0}\Gamma_1^{\alpha_1}\cdots\Gamma_7^{\alpha_7}\Box_c\varphi(t,x)$$

for a smooth function $\varphi(t,x)$ and a multi-index $\alpha=(\alpha_0,\alpha_1,\dots,\alpha_7)$. We can check that $[\Gamma_a,\Gamma_b]=\sum_{k=0}^{7} C_k^{ab}\Gamma_k$ and $[\Gamma_a,\partial_b]=\sum_{k=0}^{3} D_k^{ab}\partial_k$ with appropriate constants C_k^{ab} and D_k^{ab}. For a nonnegative integer m and a smooth function $\varphi=\varphi(t,x)$, we define

$$|\varphi(t,x)|_m=\sum_{|\alpha|\le m}|\Gamma^\alpha\varphi(t,x)|.$$

Let $x=(x_1,x_2,x_3)\in\mathbf{R}^3$. We use the polar coordinates $r=|x|$ and $\omega=(\omega_1,\omega_2,\omega_3)=x/|x|$. We put $\partial_r=\sum_{j=1}^{3}(x_j/|x|)\partial_j$, and

$$\partial_{\pm,c}=\partial_t\pm c\partial_r$$

for $c>0$. Then we have

$$\begin{aligned}\nabla_x=&\omega\partial_r-r^{-1}\omega\times\Omega,\\ \partial_t=&\frac{1}{2}(\partial_{-,c}+\partial_{+,c}),\quad \partial_r=\frac{1}{2c}(-\partial_{-,c}+\partial_{+,c}),\\ \partial_{+,c}=&\frac{1}{r}\big(cS+(r-ct)\partial_t\big)\end{aligned}$$

From these identities we obtain the following:

Lemma 4.1. *For $c>0$, there exists a positive constant C such that*

$$\begin{aligned}|\partial_t\varphi(t,x)-(-c)D_{-,c}\varphi(t,x)|+\sum_{k=1}^{3}|\partial_k\varphi(t,x)-\omega_kD_{-,c}\varphi(t,x)|&\\ \le C(1+r)^{-1}\left(|\varphi(t,x)|_1+|ct-r|\,|\partial\varphi(t,x)|\right)&\end{aligned}$$

for a smooth function $\varphi=\varphi(t,x)$, where $D_{-,c}=-(2c)^{-1}\partial_{-,c}$. We also have

$$\begin{aligned}\big|r\partial_t\varphi(t,x)-(-c)D_{-,c}\big(r\varphi(t,x)\big)\big|+\sum_{k=1}^{3}\big|r\partial_k\varphi(t,x)-\omega_kD_{-,c}\big(r\varphi(t,x)\big)\big|&\\ \le C\left(|\varphi(t,x)|_1+|ct-r|\,|\partial\varphi(t,x)|\right).&\end{aligned}$$

This lemma says that the main contribution to $\partial_t\varphi$ and $\partial_k\varphi$ $(k=1,2,3)$ are $(-c)D_{-,c}\varphi$ and $\omega_kD_{-,c}\varphi$, respectively, as far as we have a good control of $|\varphi|_1$ and $|ct-r|\,|\partial\varphi|$. As a corollary, we obtain the enhanced decay for the null forms; the key observation is that we have

$$\begin{aligned}Q_0(\varphi,\psi;c)\approx(-c)^2(D_{-,c}\varphi)(D_{-,c}\psi)-c^2|\omega|^2(D_{-,c}\varphi)(D_{-,c}\psi)=0,\\ Q_{ab}(\varphi,\psi)\approx(\omega_aD_{-,c}\varphi)(\omega_bD_{-,c}\psi)-(\omega_bD_{-,c}\varphi)(\omega_aD_{-,c}\psi)=0\end{aligned}$$

with $\omega_0=-c$, where "$\approx$" indicates the main contribution:

Corollary 4.2. *Let $c>0$, and let m be a nonnegative integer. Suppose that $Q(\varphi,\psi)=Q_0(\varphi,\psi;c)$ or $Q(\varphi,\psi)=Q_{ab}(\varphi,\psi)$. Then there is a positive constant C such that we have*

$$\begin{aligned}|Q(w_1,w_2)|_m\le&C(1+r)^{-1}\big(|\partial w|_{[m/2]}|w|_{m+1}+|w|_{[m/2]+1}|\partial w|_m\big)\\ &+C(1+r)^{-1}|r-ct|\,|\partial w|_{[m/2]}|\partial w|_m\end{aligned}$$

for any smooth function $w = (w_1, w_2)$*, where* $[m/2]$ *denotes the largest integer not exceeding* $m/2$*.*

Using the weighted L^∞–L^∞ estimates for the wave equation, and the energy inequality, Yokoyama [14] obtained the following *a priori* estimate, which leads to global existence of small solutions with the help of the local existence theorem:

Lemma 4.3. *Let* $0 < \delta \ll 1$*, and let* m *be a nonnegative integer. Suppose that the null condition is satisfied. Then for the solution* u *to* (1.1)–(1.2)*, we have*

$$|u_i(t,x)|_m \le C\varepsilon(1+t+r)^{-1}\left(\log\frac{1+c_it+r}{1+|c_it-r|}\right), \tag{4.1}$$

$$|\partial u_i(t,x)|_m \le C\varepsilon(1+r)^{-1}(1+|c_it-r|)^{-1} \tag{4.2}$$

for $i = 1, \dots, N$*, provided that* ε *is sufficiently small.*

We are in a position to observe good behavior of the null forms and the nonresonant terms of type I. We put

$$w_-(t,r) = \min_{j=1,\dots,N}(1+|c_jt-r|).$$

Let $0 < c_0 \ll 1$. If $0 \le t < c_0^{-1}$ or $0 \le r < c_0t$, we have

$$1+t+r \le C(1+|c_it-r|)$$

for $i = 1, \dots, N$ with a positive constant C, and (4.2) yields

$$|\partial u(t,x)|_m^2 \le C\varepsilon^2(1+t+r)^{-2}(1+r)^{-2}. \tag{4.3}$$

Let $r \ge c_0t \ge 1$. Then we have $1+t+r \le C(1+r)$. Hence we get

$$|\partial u_i(t,x)|_m \le C\varepsilon^2(1+t+r)^{-1}(1+|c_it-r|)^{-1}.$$

Since $c_j \ne c_k$ implies $(1+|c_jt-r|)^{-1}(1+|c_kt-r|)^{-1} \le C(1+t+r)^{-1}w_-(t,r)^{-1}$, we obtain

$$|R_i^{\mathrm{I}}(\partial u)|_m \le C\varepsilon^2(1+t+r)^{-3}w_-(t,r)^{-1}, \quad r \ge c_0t \ge 1.$$

We fix $0 < \delta \ll 1$. Then we have $\log s \le Cs^\delta$ for $s \ge 1$. Hence, in view of Corollary 4.2, it follows from (4.1) and (4.2) that

$$|N_i(\partial u)|_m \le C\varepsilon^2(1+t+r)^{-3+\delta}(1+|c_it-r|)^{-1-\delta}, \quad r \ge c_0t \ge 1.$$

Summing up we get the following lemma.

Lemma 4.4. *Fix* $0 < \delta \ll 1$*. Suppose that the null condition is satisfied, and let* u *be the solution to* (1.1)–(1.2)*. Then for a nonnegative integer* m *we have*

$$|N_i(\partial u)|_m + |R_i^{\mathrm{I}}(\partial u)|_m \le C\varepsilon^2(1+t+r)^{-2}\left((1+t+r)^{-1+\delta}w_-(t,r)^{-1-\delta} + (1+r)^{-2}\right)$$

in $[0,\infty) \times \mathbf{R}^3$ *for small* ε*.*

Now (1.16) follows immediately from Lemma 4.4, because we get

$$\|N_i(\partial u)(t)\|_{L^2(\mathbf{R}^3)} + \|R_i^{\mathrm{I}}(\partial u)(t)\|_{L^2(\mathbf{R}^3)} \le C\varepsilon^2(1+t)^{-2+\delta}.$$

By contrast we only have

$$|R_i^{\mathrm{II}}(\partial u)|_m \le C\varepsilon^2(1+r)^{-2} \sum_{\{j;c_j\neq c_i\}} (1+|c_jt-r|)^{-2}, \tag{4.4}$$

which just gives $\|R_i^{\mathrm{II}}(\partial u)(t)\|_{L^2(\mathbf{R}^3)} \le C\varepsilon^2(1+t)^{-1}$.

In [5] the author studied the Cauchy problem for

$$\Box_{c_i} u_i = F_i(u, \partial u), \quad i = 1, \dots, N$$

in three space dimensions, where $F_i = N_i(\partial u) + R_i^{\mathrm{I}}(\partial u) + R_i^{\mathrm{III}}(u, \partial u)$ with a new kind of nonlinearity

$$R_i^{\mathrm{III}}(u, \partial u) = \sum_{\{j,k;\, c_j\neq c_k\}} \sum_{a=0}^{3} B_{ijk}^a u_j(\partial_a u_k)$$

with some constants B_{ijk}^a. We refer to R_i^{III} as the nonresonant terms of type III. Global existence of small solutions under the null condition (1.14) is proved for the system above (see [8] for further development). Employing the *a priori* estimates obtained in [5] we get the following.

Lemma 4.5. *Let* $0 < \rho < 1$, *and let* m *be a nonnegative integer. Suppose that the null condition is satisfied, and* $R_i^{\mathrm{II}}(\partial u) \equiv 0$ *for all* $i = 1, \dots, N$. *Then for the solution* u *to* (1.1)–(1.2), *we have*

$$|u_i(t,x)|_m \le C\varepsilon(1+t+r)^{-1}(1+|c_it-r|)^{-1}, \tag{4.5}$$

$$|\partial u_i(t,x)|_m \le C\varepsilon(1+r)^{-1}(1+|c_it-r|)^{-1-\rho} \tag{4.6}$$

for $i = 1, \dots, N$, *provided that* ε *is sufficiently small.*

This is the reason why we have the improvement in Theorem 3.3 compared to Theorem 3.1.

Remark 4.6. Knowing the global existence results in [14] and [5], one may expect that SDGE holds for the system

$$\Box_{c_i} u_i = F_i(u, \partial u) := N_i(\partial u) + R_i^{\mathrm{I}}(\partial u) + R_i^{\mathrm{II}}(\partial u) + R_i^{\mathrm{III}}(u, \partial u), \quad 1 \le i \le N$$

when the null condition (1.14) is satisfied, because SDGE for the case $R_i^{\mathrm{III}} \equiv 0$ is proved in [14], and SDGE for the case $R_i^{\mathrm{II}} \equiv 0$ in [5]. However this expectation turns out to be false in general; Ohta [11] proved that SDGE fails for the system

$$\begin{cases} \Box_{c_1} u_1 = F_1(u, \partial u) := u_2(\partial_t u_1), \\ \Box_{c_2} u_2 = F_2(u, \partial u) := (\partial_t u_1)^2 \end{cases}$$

when $0 < c_1 < c_2$. Observe that F_1 and F_2 are nonresonant terms of types III and II, respectively.

5. Outline of the Proof for Theorems 3.1 and 3.3

Finally we would like to explain the main tool to obtain the asymptotic pointwise behavior. Switching to the polar coordinates, we have

$$\partial_{+,c}\big(D_{-,c}(r\varphi)\big) = -(2c)^{-1}\left(r\Box_c\varphi + r^{-1}\Delta_\omega\varphi\right) \tag{5.1}$$

for a smooth function φ, where $\Delta_\omega = \sum_{j=1}^{3}\Omega_j^2$. We use this equation to obtain the asymptotic pointwise behavior by applying Proposition 5.1 below. Then with the help of Lemma 4.1 we get the asymptotic pointwise behavior for ∂u.

We choose $0 < c_0 \ll 1$ and set

$$\Lambda_0 = \{(t,r) \in (0,\infty)\times(0,\infty); r \geq c_0 t \geq 1\}.$$

For $i = 1, \dots, N$, we put $t_{0,i}(\sigma) = \max\{-\sigma/(c_i - c_0), 1/c_0\}$.

Proposition 5.1. *Let $c_1, \dots, c_N$ be positive constants. Fix arbitrary $i \in \{1, \dots, N\}$. Suppose that $\mu_1, \dots, \mu_N$ and $\kappa_1, \dots, \kappa_N$ are nonnegative constants satisfying the following:*

- *$\mu_j > 1$ and $\kappa_j \geq 0$ if $c_j = c_i$.*
- *$\mu_j \geq 0$ and $\kappa_j > 1$ if $c_j \neq c_i$.*

If we have

$$\partial_{+,c_i}v(t,r,\omega) = G(t,r,\omega), \quad (t,r)\in\Lambda_0,\ \omega\in S^2,$$

and

$$|G(t,r,\omega)| \leq \sum_{j=1}^{N} B_j(1+t+r)^{-\mu_j}(1+|c_jt-r|)^{-\kappa_j}, \quad (t,r)\in\Lambda_0,\ \omega\in S^2 \tag{5.2}$$

with some nonnegative constants $B_1, \dots, B_N$, then there exists a positive constant C such that

$$\begin{aligned}|v(t,r,\omega) - V(r-c_it,\omega)| \leq & C\sum_{\{j;c_j=c_i\}} B_j(1+t+r)^{-\mu_j+1}(1+|c_jt-r|)^{-\kappa_j} \\ & + \sum_{\{j;c_j\neq c_i\}} B_j(1+t+r)^{-\mu_j}\end{aligned}$$

for any $(t,r)\in\Lambda_0$ and $\omega\in S^2$, where

$$V(\sigma,\omega) = v\big(t_{0,i}(\sigma), r_{0,i}(\sigma),\omega\big) + \int_{t_{0,i}(\sigma)}^{\infty} G(s,\sigma+c_is,\omega)ds$$

and $r_{0,i}(\sigma) = \sigma + c_it_{0,i}(\sigma)$. Here the constant C may depend on μ_j and κ_j, but are independent of B_j.

Proof. The proof is similar to that of Lemma 2.1 in a sense: We can explicitly solve v and we obtain

$$|v(t,r,\omega) - V(r-c_it,\omega)| \leq \int_t^{\infty} \big|G\big(s, r-c_i(t-s),\omega\big)\big|\,ds. \tag{5.3}$$

We use (5.2) to estimate the right-hand side of (5.3). We omit the details of calculation here. $\square$

Observe that the difference of the propagation speeds plays important role in the assumption of Proposition 5.1.

Outline of proof of Theorem 3.1. We set $\left(u_i^0(t,\cdot), \partial_t u_i^0(t,\cdot)\right) = S_{c_i}(t)(f_i, g_i)$, and $\widetilde{u}_i = u_i - \varepsilon u_i^0$. By Lemma 2.2, we know that the asymptotic pointwise behavior for $\partial_a u_i^0(t,x)$ can be written in terms of $\partial_\sigma U_i^0(\sigma,\omega)$. Hence our task is to obtain the asymptotic behavior for $\partial_a \widetilde{u}_i$ for $0 \le a \le 3$.

We set

$$\widetilde{v}_i(t,r,\omega) = D_{-,c_i}\left(r\widetilde{u}_i(t,r\omega)\right).$$

Then we have

$$\partial_{+,c_i}\widetilde{v}_i(t,r,\omega) = R_i(t,r,\omega),$$

where

$$R_i(t,r,\omega) = -(2c_i)^{-1}\left(rF_i\big(\partial u(t,r\omega)\big) + r^{-1}\Delta_\omega \widetilde{u}_i(t,r\omega)\right)$$

(cf. (5.1)). It follows from Lemma 4.4 and (4.4) that

$$\begin{aligned}\left|rF_i\big(\partial u(t,r\omega)\big)\right| \le & \sum_{\{j;c_j=c_i\}} C\varepsilon^2(1+t+r)^{-2+\delta}(1+|c_jt-r|)^{-1-\delta}\\ & + \sum_{\{j;c_j\neq c_i\}} C\varepsilon^2(1+t+r)^{-1}(1+|c_jt-r|)^{-2}\end{aligned}$$

for $(t,r) \in \Lambda_0$ and $\omega \in S^2$. Similarly to (4.1), we get

$$|r^{-1}\Delta_\omega\widetilde{u}_i(t,r\omega)| \le C\varepsilon^2(1+t+r)^{-2+\delta}(1+|c_it-r|)^{-\delta}, \quad (t,r)\in\Lambda_0,\ \omega\in S^2.$$

Hence if we put

$$\varepsilon\widetilde{V}_i(\sigma,\omega) = \widetilde{v}_i\big(t_{0,i}(\sigma), r_{0,i}(\sigma),\omega\big) + \int_{t_{0,i}(\sigma)}^{\infty} R_i(s,\sigma+c_is,\omega)\,ds,$$

then Lemma 5.1 implies

$$(5.4) \qquad |\widetilde{v}_i(t,r,\omega) - \varepsilon\widetilde{V}_i(r-c_it,\omega)| \le C\varepsilon^2(1+t+r)^{-1+\delta}(1+|c_it-r|)^{-\delta}$$

for $(t,r)\in\Lambda_0$ and $\omega \in S^2$. Finally we set $V_i(\sigma,\omega) = U_i^0(\sigma,\omega) + \widetilde{V}_i(\sigma,\omega)$. Using Lemmas 4.1 and 4.3, we obtain (3.1) and (3.2) from (5.4). □

Outline of proof of Theorem 3.3. Going similar lines to the proof of Theorem 3.1, but using Lemma 4.5 instead of Lemma 4.3, we can easily obtain (3.7) and (3.8) for $k=1$. We use (3.7) to obtain

$$\begin{aligned}|ru_i(t,r,\omega) - \varepsilon U_i(r-c_it,\omega)| \le & \left|\int_r^\infty \partial_\lambda\left(\lambda u_i(t,\lambda\omega) - \varepsilon U_i(\lambda - c_it,\omega)\right)d\lambda\right|\\ \le & C\varepsilon\int_r^\infty (1+t+\lambda)^{-1}(1+|c_it-\lambda|)^{-\rho}d\lambda,\end{aligned}$$

which yields (3.6). (3.8) for $k=0$ can be similarly obtained from (3.8) for $k=1$. This completes the proof. □

6. Outline of the Proof of Theorem 3.4

Before we state the outline of the proof of Theorem 3.4, we give a simple proof of the blow up of solutions for (1.3), following the argument in Godin [2] where two space-dimensional case was treated (see also [7]), because the necessary tools are similar.

Proposition 6.1. *Consider the Cauchy problem*

$$\square u(t,x) = \big(\partial_t u(t,x)\big)^2, \qquad (t,x) \in [0,T) \times \mathbf{R}^3, \tag{6.1}$$

$$u(0,x) = 0, \ (\partial_t u)(0,x) = \varepsilon g(x), \qquad x \in \mathbf{R}^3. \tag{6.2}$$

If $g(\not\equiv 0) \in C_0^\infty(\mathbf{R}^3)$ is radially symmetric and nonnegative, then for any $\varepsilon > 0$ there is a positive and finite time T_ε such that

$$\lim_{t \nearrow T_\varepsilon} \sup_{x \in \mathbf{R}^3} |(\partial_t u)(t,x)| = \infty.$$

and we cannot extend the classical solution u beyond the time T_ε.

Proof. Let $g(x) = g^*(|x|)$ with some function g^*. Then we can show that u is radially symmetric in x-variable, namely there exists a function $u^* = u^*(t,r)$ such that $u(t,x) = u^*(t,|x|)$. We put $\breve{g}(r) = rg^*(|r|)$ for $r \in \mathbf{R}$, and $\breve{u}(t,r) = ru^*(t,|r|)$ for $(t,r) \in [0,\infty) \times \mathbf{R}$. Then we have

$$\breve{u}(t,r) = \frac{\varepsilon}{2c}\int_{r-ct}^{r+ct} \breve{g}(\lambda)d\lambda + \frac{1}{2c}\int_0^t \left(\int_{r-c(t-\tau)}^{r+c(t-\tau)} \frac{1}{\lambda}(\partial_t \breve{u})^2(\tau,\lambda)d\lambda \right) d\tau, \tag{6.3}$$

and

$$\begin{aligned} \partial_t \breve{u}(t,r) =& \varepsilon\frac{\breve{g}(r-ct)+\breve{g}(r+ct)}{2} + \frac{1}{2}\int_0^t \frac{(\partial_t\breve{u})^2\big(\tau, r+c(t-\tau)\big)}{r+c(t-\tau)}d\tau \\ &+ \frac{1}{2}\int_0^t \frac{(\partial_t\breve{u})^2\big(\tau, r-c(t-\tau)\big)}{r-c(t-\tau)}d\tau. \end{aligned} \tag{6.4}$$

We suppose that $g^*(\lambda) \geq 0$ for all $\lambda \geq 0$. Let $r - ct = \sigma > 0$. Then, since $\breve{g}(r+ct) \geq 0$, we get

$$\partial_t \breve{u}(t, ct+\sigma) \geq \frac{\varepsilon \breve{g}(\sigma)}{2} + \frac{1}{2}\int_0^t \frac{(\partial_t \breve{u})^2(\tau, c\tau+\sigma)}{c\tau+\sigma}d\tau =: W(t,\sigma).$$

We obtain

$$\partial_t W(t,\sigma) = \frac{(\partial_t\breve{u})^2(t,ct+\sigma)}{2(ct+\sigma)} \geq \frac{W^2(t,\sigma)}{2(ct+\sigma)},$$

which leads to

$$\begin{aligned} \partial_t \breve{u}(t,ct+\sigma) \geq W(t,\sigma) \geq& \frac{2cW(0,\sigma)}{2c - W(0,\sigma)\log(c\sigma^{-1}t+1)} \\ =& \frac{2c\varepsilon\breve{g}(\sigma)}{4c - \varepsilon\breve{g}(\sigma)\log(c\sigma^{-1}t+1)}, \end{aligned}$$

whose right-hand side blows up to infinity as $t \nearrow c^{-1}\sigma\left(\exp\left(4c/(\varepsilon\breve{g}(\sigma))\right)-1\right)$ if $\breve{g}(\sigma)>0$. This completes the proof. □

Now we turn our attention to the outline of the proof of Theorem 3.4. We suppose that $g=(g_1,g_2)$ is nonnegative and radially symmetric, and let u be the solution to (3.9) with initial data $u=0$ and $\partial_t u=\varepsilon g$ at $t=0$. We assume $0<c_2<c_1$. For $i=1,2$, we write $g_i(x)=g_i^*(|x|)$ and $u_i(t,|x|)=u_i^*(t,|x|)$. We put $\breve{g}_i(r)=rg_i^*(|r|)$ for $r\in\mathbf{R}$, and $\breve{u}_i(t,r)=ru_i^*(t,|r|)$ for $(t,r)\in[0,\infty)\times\mathbf{R}$.
From a similar formula to (6.4), we get

$$\partial_t\breve{u}_1(t,r)\geq\varepsilon\frac{\breve{g}_1(r-c_1t)}{2}\geq 0 \tag{6.5}$$

for $r\geq c_1t$. Similarly to (6.3), we obtain

$$\begin{aligned}\breve{u}_2(t,r)=&\frac{\varepsilon}{2c_2}\int_{r-c_2t}^{r+c_2t}\breve{g}_2(\lambda)d\lambda\\&+\frac{1}{2c_2}\int_0^t\left(\int_{r-c_2(t-\tau)}^{r+c_2(t-\tau)}\frac{1}{\lambda}(\partial_t\breve{u}_1)^2(\tau,\lambda)d\lambda\right)d\tau.\end{aligned}\tag{6.6}$$

From (6.6) the upper bound in (3.10) is easily obtained (see also (4.1)). To obtain the lower bound, we assume that $0\leq r-c_2t=\sigma\leq R$. If t is large enough, we have

$$\begin{aligned}&\left\{(\tau,\lambda);\ \frac{\sigma}{c_1-c_2}\leq\tau\leq\frac{2c_2t+\sigma-R}{c_1+c_2},\ c_1\tau\leq\lambda\leq c_1\tau+R\right\}\\&\quad\subset\{(\tau,\lambda);\ 0\leq\tau\leq t,\ r-c_2(t-\tau)\leq\lambda\leq r+c_2(t-\tau)\}.\end{aligned}$$

Hence it follows from (6.5) and (6.6) that

$$\begin{aligned}\breve{u}_2(t,c_2t+\sigma)\geq&\frac{\varepsilon}{2c_2}\int_\sigma^{2c_2t+\sigma}\breve{g}_2(\lambda)d\lambda\\&+\frac{\varepsilon^2}{8c_2}\int_{\frac{\sigma}{c_1-c_2}}^{\frac{2c_2t+\sigma-R}{c_1+c_2}}\left(\int_{c_1\tau}^{c_1\tau+R}\lambda^{-1}|\breve{g}_1(\lambda-c_1\tau)|^2d\lambda\right)d\tau\\\geq&\frac{\varepsilon}{2c_2}\int_\sigma^{2c_2t+\sigma}\breve{g}_2(\lambda)d\lambda\\&+\frac{\varepsilon^2}{8c_2}\left(\int_0^R|\breve{g}_1(\lambda)|^2d\lambda\right)\int_{\frac{\sigma}{c_1-c_2}}^{\frac{2c_2t+\sigma-R}{c_1+c_2}}\frac{1}{c_1\tau+R}d\tau\end{aligned}\tag{6.7}$$

for large t, which implies the desired lower bound if we choose appropriate g_1 and g_2. □

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